Contact stationary Legendrian surfaces and Legendrian Willmore surfaces in S^5

Yong Luo

Abstract

Let $(M^5, \alpha, g_\alpha, J)$ be a 5-dimensional Sasakian Einstein manifold with contact 1-form α, associated metric g_α and the almost complex structure J and assume that L is a contact stationary Legendrian surface in M^5. We prove that L satisfies the following equation

$$-\Delta^\nu H + KH = 0,$$

where Δ^ν is the normal Laplacian w.r.t the metric g on L induced from g_α and K is the Gauss curvature of (L, g).

Using this equation and a new Simons type inequality for Legendrian surfaces in the standard sphere S^5, we prove an integral inequality for contact stationary Legendrian surfaces in S^5. In particular we prove that if L is a Legendrian stationary surface in S^5, B is the second fundamental form of L, $S = |B|^2$ and

$$0 \leq S \leq 2,$$

then we have either $S = 0$ and L is totally geodesic or $S = 2$ and L is a flat Legendrian minimal torus.

In addition, by using a similar Simons type inequality (inequality (3.32)) we prove that if L is a Legendrian Willmore surface in S^5 and

$$0 \leq S \leq 2,$$

then we have either $S = 0$ and L is totally geodesic or $S = 2$ and L is a flat Legendrian minimal torus.

1 Introduction

Let $(M^{2n+1}, \alpha, g_\alpha, J)$ be a $2n+1$ dimensional contact metric manifold with contact structure α, associated metric g_α and almost complex structure J. Assume that (L, g) is an n-diemsional compact Legendrian submanifold of M^{2n+1} with metric g induced from g_α. The volume of L is defined by

$$V(L) = \int_N d\mu, \tag{1.1}$$

where $d\mu$ is the volume form of g.

A contcat stationary Legendrian submanifold of M^{2n+1} is a Legendrian submanifold of M^{2n+1} which is a stationary point of V w.r.t. Legendrian deformations. That is we call a

1

Legendrian submanifold $L \subseteq M^{2n+1}$ a contact stationary Legendrian submanifold, if for any Legendrian deformations $L_t \subseteq M^{2n+1}$ with $L_0 = L$ we have

$$\frac{dV(L_t)}{dt}\big|_{t=0} = 0.$$

The E-L equation for a contact stationary Legendrian submanifold L is

$$div_g(JH) = 0, \tag{1.2}$$

where div_g is the divergence w.r.t g and H is the mean curvature vector of L in M^{2n+1}.

Corresponding geometrically constrained variational problem for the area functional in Kähler geometry was firstly proposed by Oh ([Oh93]). In his paper he proposed the conception of H-minimal Lagrangian submanifolds (Lagrangian submanifolds which are critical points of the area functional under Hamiltonian deformations) in Kähler manifolds. Later in [ScW] Schoen and Wolfson considered the area restricted to the Lagrangian surfaces in a Kähler surface, or more generally a symplectic 4-dimensional manifold with suitable metric. They studied its critical points and in particular its minimizers and applied their study to the problem of finding canonical representatives of the Lagrangian homology. Legendrian submanifolds in a Sasakian manifold M^{2n+1} can be seem as links of Lagrangian submanifolds in the cone CM^{2n+1} which is a Kähler manifold with proper metric and complex structure (see section 2). Therefore variational problem of the area restricted to Legendrian submanfolds in a Sasakian manifold is a nature analogue of the related problem in Kähler geometry.

From the definition we see that Legendiran minimal surfaces are a special kind of contact stationary Legendrian surfaces. Another special kind of contact stationary Legendrian surfaces are Legendrian surfaces with parallel mean curvature vector field in the normal bundle. The study of contact stationary Legendrian submanifolds of S^{2n+1} is relatively recent endeavor. For $n = 1$, contact stationary Legendrian curves are the so called (p, q) curves discovered by Schoen and Wolfson in [ScW]. For $n = 2$, since harmonic 1-form on a 2-sphere must be trial, contact stationary Legendrian 2-sphere must be minimal and so must be equatorial 2-spheres by Yau's result ([Yau]). There are a lot of contact stationary doubly periodic surfaces form R^2 to S^5 by lifting Hélein and Romon's examples ([HR]). For $n \geq 3$ case, very little is known. But general constructions are given by Castro, Li and Urbaro in [C-Li-U] and by Butscher in [Bu] . For more complete information on the existence construction of Legendrian minimal submanifolds and contact stationary Legendrian submanifolds we refer to a nice survey by Butscher ([Bu]) and references therein.

One of the targets of this paper is to study pinching properties of contact stationary Legendrian surfaces in S^5. To do this we first prove an equation satisfied by contact stationary Legendrian surfaces in a Sasakian Einstein manifold, which we hope will be useful in analyzing analytic properties of contact stationary Legendrian surfaces.

Theorem 1.1. *Let L be a contact stationary Legendrian surface in a 5-dimensional Sasakian Einstein manifold $(M^5, \alpha, g_\alpha, J)$, then L satisfies the following equation:*

$$-\Delta^\nu H + KH = 0, \tag{1.3}$$

where Δ^ν is the normal Laplacian w.r.t the metric g on L induced from g_α and K is the Gauss curvature of (L, g).

We see this equation is similar to the well-known Willmore surface equation, i.e. they have the same principle part and their nonlinear terms have the same critical order. For the Willmore surface equation, see (1.8).

We recall that the well-known Clifford torus is

$$T_{Clif} = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}). \tag{1.4}$$

In the theory of minimal surfaces, the following Simons integral inequality and Pinching theorem due to Simons ([Siom]), Lawson ([Law]) and Chern et al. ([C-C-K]) are well-known.

Theorem 1.2 (Simons, Lawson, Chern-Do Carmo-Kobayashi). *Let M be a compact minimal surface in a unit sphere S^3 and B is the second fundamental form of M in S^3. Set $S = |B|^2$, then we have*

$$\int_M S(2 - S)d\mu \leq 0.$$

In particular, if

$$0 \leq S \leq 2,$$

then either $S = 0$ and M is totally geodesic, or $S = 2$ and M is the Clifford torus T_{Clif}, which is defined by (1.4).

The above integral inequality was proved by Simons in his celebrated paper [Siom] and the classification result was given by Chern et al. ([C-C-K]) and Lawson ([Law]), independently.

For minimal surfaces in a sphere with higher codimension, corresponding integral inequality was proved by Benko et al. ([BKSS]) or Kozlowski et al. ([KoSi]). In order to state their result, we first record an example.

Example. The veronese surface is a minimal surface in $S^4 \subseteq R^5$ defined by

$$u : S^2(\sqrt{3}) \subseteq R^3 \quad \mapsto \quad S^4(1) \subseteq R^5$$
$$(x, y, z) \quad \mapsto \quad (u_1, u_2, u_3, u_4, u_5)$$

where

$$u_1 = \frac{1}{\sqrt{3}}yz, u_2 = \frac{1}{\sqrt{3}}xz, u_3 = \frac{1}{\sqrt{3}}xy,$$
$$u_4 = \frac{1}{2\sqrt{3}}(x^2 - y^2), u_5 = \frac{1}{6}(x^2 + y^2 - 2z^2).$$

u defines an isometric immersion of $S^2(\sqrt{3})$ into $S^4(1)$, and it maps two points (x, y, z), $(-x, -y, -z)$ of $S^2(\sqrt{3})$ into the same point of $S^4(1)$, and so it imbedded the real projective plane into $S^4(1)$.

Theorem 1.3 ([BKSS], [KoSi]). *Let M be a minimal surface in an n-dimensional sphere S^n. Then we have*

$$\int_M S(2 - \frac{3}{2}S)d\mu \leq 0. \tag{1.5}$$

In particular, if

$$0 \leq S \leq \frac{4}{3},$$

then either $S = 0$ and M is totally geodesic, or $S = \frac{4}{3}$,n=4 and M is the Veronese surface.

The above classification for minimal surfaces in a sphere with $S = \frac{4}{3}$ was also got by Chern et al. in [C-C-K].

We see that the (first) pinching constant for minimal surfaces in S^3 is 2, but it is $\frac{4}{3}$ for minimal surfaces of higher codimensions. This is an interesting phenomenon and we think this dues to the complexity of the normal bundle, because for Legendrian minimal surfaces in S^5, the (first) pinching constant is also 2.

Theorem 1.4 ([YKM]). *If M is a Legendrian minimal surface of the unit sphere S^5 and $0 \leq S \leq 2$, then S is identically 0 or 1.*

Remark 1.5. *This pinching theorem is optimal, by virtual of a result due to Haskins ([Ha], theorem 5.5).*

For higher dimension case of this theorem we refer to [DV].

All of these results are based on calculating the Laplacian of S and then get Simons type equalities or inequalities. The minimal condition is used to cancel some terms in the resulting calculation and to some extent it is important. In this note we prove a Simons type inequality (lemma 3.7) for Legendrian surfaces in S^5, without minimal condition. By using equation (1.3) and this Simons type inequality we get

Theorem 1.6. *Let $L : \Sigma \mapsto S^5$ be a contact stationary Legendrian surface, where S^5 is the unit sphere with standard contact structure and metric (as given in the end of section 2). Then we have*

$$\int_L \frac{3}{2}\rho^2(2 - S) + 2H^2\rho^2 + 2H^2 d\mu \leq 0,$$

where $\rho^2 := S - 2H^2$. In particular, if

$$0 \leq S \leq 2,$$

then either $S = 0$ and L is totally geodesic, or $S = 2$ and L is a flat Legendrian minimal torus.

Because Legendrian minimal surfaces are contact stationary Legendrian surfaces we see that theorem 1.4 is a corollary of theorem 1.6.

As we said before, the equation (1.3) satisfied by contact stationary Legendrian surfaces is quite similar to the Willmore surface equation. Hence we could also prove corresponding integral inequality and pinching theorem for Legendrian Willmore surfaces in S^5. To state this theorem, we first recall some material from the Willmore functional theory.

Let M be a closed surface in the n-dimensional unit sphere S^n with metric g induced from S^n. The Willmore energy of M is defined by

$$W(M) = \int_M \rho^2 d\mu, \tag{1.6}$$

4

where $\rho^2 = S - 2H^2$ and $d\mu$ is the area measure element w.r.t the metric g.

We see that the Willmore energy is nonnegative and it equals to zero only at totally umbilical surfaces.

A surface M is called a Willmore surface if and only if it is a critical point of the Willmore functional. It satisfies the following E-L equation ([Weiner]):

$$\Delta^\nu H^\alpha + \sum_{i,j,\beta} h_{ij}^\alpha h_{ij}^\beta H^\beta - 2H^2 H^\alpha = 0, 3 \le \alpha \le n, \tag{1.7}$$

or equivalently,

$$\Delta^\nu H + Q(A^\circ)H = 0, \tag{1.8}$$

where $Q(A^\circ)H = g^{ik}g^{jl}A_{ij}^\circ \langle A_{kl}^\circ, H \rangle$, and A° is the trace free part of A.

For Willmore surfaces in a sphere, Li proved two integral inequalities and pinching theorems, which are generalizations of theorem 1.2 and theorem 1.3.

Theorem 1.7 ([Li01])**.** *Let M be a compact Willmore surface in S^3. Then we have*

$$\int_M \rho^2(2 - \rho^2)d\mu \le 0. \tag{1.9}$$

In particular, if

$$0 \le \rho^2 \le 2,$$

then either $\rho^2 = 0$ and M is totally umbilical, or $\rho^2 = 2$ and M is the Clifford torus.

Theorem 1.8 ([Li02])**.** *Let M be a compact Willmore surface in S^n. Then we have*

$$\int_M \rho^2(2 - \frac{3}{2}\rho^2)d\mu \le 0. \tag{1.10}$$

In particular, if

$$0 \le \rho^2 \le \frac{4}{3},$$

then either $\rho^2 = 0$ and M is totally umbilical, or $\rho^2 = \frac{4}{3}$, n=4 and M is the Veronese surface.

Once again we see that the (first) pinching constants for Willmore surfaces of codimension 1 and higher codimensions are different. But we prove that Legendrian Willmore surfaces in S^5, though with higher codimension, have the (first) pinching constant 2.

Theorem 1.9. *Let L be a Legendrian Willmore surface in S^5. Then we have*

$$\int_L (\rho^2 + \frac{S}{2})(2 - S)d\mu \le 0, \tag{1.11}$$

Furthermore if

$$0 \le S \le 2,$$

then either $S = 0$, i.e. L is totally geodesic or $S = 2$ and L is a flat Legendrian minimal torus.

Because Legendrian minimal surfaces are Legendiran Willmore surfaces, we see that theorem 1.4 is also a corollary of theorem 1.9.

We end this introduction by recalling a classification theorem for flat Legendrian minimal toruses in S^5. For a constant θ let T_θ be the 2-torus in S^5 defined by

$$T_\theta = \{(z_1, z_2, z_3) \in C^3 : |z_i| = \frac{1}{3}, i = 1, 2, 3 \ \ and \ \ \sum_i arg z_i = \theta\}.$$

T_θ is called the generalized Clifford torus and it is a flat Legendrian minimal torus in S^5. Its projection under the Hopf map $\pi : S^5 \mapsto CP^2$ is a flat minimal Lagrangian torus , which is also called a generalized Clifford torus. It is proved in [LOY] that a flat Lagrangian minimal torus in CP^2 must be $S^1 \times S^1$. By the correspondence of Lagrangian minimal surfaces in CP^2 and Legendrian minimal surfaces in S^5 (see [Rec]) we see that a flat Legendrian minimal torus in S^5 must be a generalized Clifford torus. For more details we refer to [Ha], page 853.

The rest of this paper is organized as follows: In section 2 we collect some basic material from Sasakian geometry, which will be used in the next section. In section 3 we prove our main results, theorem 1.1, theorem 1.6 and theorem 1.9.

Acknowledgement. I would like to thank my advisor, Professor Guofang Wang for a lot of discussion on Sasakian geometry and useful suggestion during the preparation of this paper. After I posted the first version of this paper to arXiv.org I received useful comments and suggestion which help to improve this paper from professor Ildefonso Castro.

2 Preliminaries on contact geometry

In this section we recall some basic material from contact geometry. For more information we refer to [Blair].

2.1 Contact Manifolds

Definition 2.1. *A contact manifold M is an odd dimensional manifold with a one form α such that $\alpha \wedge (d\alpha)^n \neq 0$, where $dim M = 2n + 1$.*

Assume now that (M, α) is a given contact manifold of dimension $2n + 1$. Then α defines a 2n-dimensional vector bundle over M, where the fibre at each point $p \in M$ is given by

$$\xi_p = Ker \alpha_p.$$

Sine $\alpha \wedge (d\alpha)^n$ defines a volume form on M, we see that

$$\omega := d\alpha$$

is a closed nondegenerate 2-form on $\xi \oplus \xi$ and hence it defines a symplectic product on ξ such that $(\xi, \omega|_{\xi \oplus \xi})$ becomes a symplectic vector bundle. A consequence of this fact is that there exists an almost complex bundle structure

$$\tilde{J} : \xi \mapsto \xi$$

compatible with $d\alpha$, i.e. a bundle endomorphism satisfying:
(1) $\tilde{J}^2 = -id_\xi$,

(2) $d\alpha(\tilde{J}X, \tilde{J}Y) = d\alpha(X, Y)$ for all $X, Y \in \xi$,

(3) $d\alpha(X, \tilde{J}X) > 0$ for $X \in \xi \setminus 0$.

Since M is an odd dimensional manifold, ω must be degenerate on TM, and so we obtains a line bundle η over M with fibres

$$\eta_p := \{V \in T_p M | \omega(V, W) = 0 \;\; \forall \;\; W \in \xi_p\}.$$

Definition 2.2. *The Reeb vector field $\boldsymbol{R}$ is the section of η such that $\alpha(\boldsymbol{R}) = 1$.*

Thus α defines a splitting of TM into a line bundle η with the canonical section $\mathbf{R}$ and a symplectic vector bundle $(\xi, \omega|\xi \oplus \xi)$. We denote the projection along η by π, i.e.

$$\pi : TM \mapsto \xi,$$
$$\pi(V) := V - \alpha(V)\mathbf{R}.$$

Using this projection we extend the almost complex structure $\tilde{J}$ to a section $J \in \Gamma(T^*M \otimes TM)$ by setting

$$J(V) = \tilde{J}(\pi(V)),$$

for $V \in TM$.

We call J an almost complex structure of the contact manifold M.

Definition 2.3. *Let (M, α) be a contact manifold, a submanifold L of (M, α) is called an isotropic submanifold if $T_x L \subseteq \xi_x$ for all $x \in L$.*

For algebraic reasons the dimension of an isotropic submanifold of a $2n + 1$ dimensional contact manifold can not be bigger than n.

Definition 2.4. *An isotropic submanifold $L \subseteq (M, \alpha)$ of maximal possible dimension n is called a Legendrian submanifold.*

2.2 Sasakian manifolds

Let (M, α) be a contact manifold, with almost complex structure J and Reeb field $\mathbf{R}$. A Riemannian metric g_α defined on M is said to be associated, if it satisfies the following three conditions:

(1) $g_\alpha(\mathbf{R}, \mathbf{R}) = 1$,

(2) $g_\alpha(V, \mathbf{R}) = 0, \forall \;\; V \in \xi$,

(3) $\omega(V, JW) = g_\alpha(V, W), \forall \;\; V, W \in \xi$.

We should mention here that on any contact manifold there exists an associated metric on it, because we can construct one in the following way. We introduce a bilinear form b by

$$b(V, W) := \omega(V, JW),$$

then the tensor

$$g := b + \alpha \otimes \alpha$$

defines an associated metric on M.

Sasakian manifolds are the odd dimensional analogue of Kähler manifolds. They are defined as follows.

Definition 2.5. *A contact manifold (M, α) with an associated metric g_α is called Sasakian, if the cone CM equipped with the following extended metric $\bar{g}$*

$$(CM, \bar{g}) = (R_+ \times M, dr^2 + r^2 g_\alpha) \tag{2.1}$$

is Kähler w.r.t the following canonical almost complex structure J on $TCM = R \oplus \langle \boldsymbol{R} \rangle \oplus \xi :$

$$J(r\partial r) = \boldsymbol{R}, J(\boldsymbol{R}) = r\partial r.$$

Furthermore if g_α is Einstein, M is called a Sasakian Einstein manifold.

We record several lemmas which are well known in Sasakian geometry. These lemmas will be used in the next section.

Lemma 2.6. *Let (M, α, g_α, J) be a Sasakian manifold. Then*

$$\bar{\nabla}_X \boldsymbol{R} = -JX, \tag{2.2}$$

and

$$(\bar{\nabla}_X J)(Y) = g(X, Y)\boldsymbol{R} - \alpha(Y)X, \tag{2.3}$$

for $X, Y \in TM$, where $\bar{\nabla}$ is the Levi-Civita connection on (M, g_α).

Lemma 2.7. *Let L be a Legendrian submanifold in a Sasakian Einstein manifold (M, α, g_α, J), then the mean curvature form $\omega(H, \cdot)|_L$ defines a closed one form on L.*

For a proof of this lemma we refer to [Le], Proposition A.2 or [Smoc], lemma 2.8. In fact they proved this result under a weaker assumption that (M, α, g_α, J) is a weakly Sasakian Einstein manifold, where weakly Einstein means that g_α is Einstein only when restricted to the contact hyperplane $Ker\alpha$.

Lemma 2.8. *Let L be a Legendrian submanifold in a Sasakian manifold (M, α, g_α, J) and A be the second fundamental form of L in M. Then we have*

$$g_\alpha(A(X, Y), \boldsymbol{R}) = 0, \tag{2.4}$$

for any $X, Y \in TL$.

Proof. For any $X, Y \in TL$,

$$\begin{aligned}
\langle A(X, Y), \mathbf{R} \rangle &= \langle \bar{\nabla}_X Y, \mathbf{R} \rangle \\
&= -\langle Y, \bar{\nabla}_X \mathbf{R} \rangle \\
&= \langle Y, JX \rangle \\
&= 0,
\end{aligned}$$

where in the third equality we used (2.2). $\square$

In particular this lemma implies that the mean curvature H of L is orthogonal to the Reeb field $\mathbf{R}$.

Lemma 2.9. *For any $Y, Z \in Ker\alpha$, we have*

$$g_\alpha(\bar{\nabla}_X(JY), Z) = g_\alpha(J\bar{\nabla}_X Y, Z). \tag{2.5}$$

Proof. Note that

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X(JY) - J\bar{\nabla}_X Y.$$

Therefore by using (2.3) we have

$$
\begin{aligned}
\langle \bar{\nabla}_X(JY), Z \rangle &= \langle (\bar{\nabla}_X J)Y, Z \rangle - \langle J\bar{\nabla}_X Y, Z \rangle \\
&= -\langle J\bar{\nabla}_X Y, Z \rangle,
\end{aligned}
$$

for any $Y, Z \in Ker\alpha$. $\qquad\square$

A most canonical example of Sasakian Einstein manifolds is the standard odd dimensional sphere S^{2n+1}.

The standard sphere S^{2n+1}. Let $C^n = R^{2n+2}$ be the Euclidean space with coordinates $(x_1, y_1, ..., x_{n+1}, y_{n+1})$ and S^{2n+1} be the standard unit sphere in R^{2n+2}. Define

$$\alpha_0 = \sum_{j+1}^{n+1}(x_j dy_j - y_j dx_j),$$

then

$$\alpha := \alpha_0\big|_{S^{2n+1}}$$

defines a contact one form on S^{2n+1}. Assume that g_0 is the standard metric on R^{2n+2} and J_0 is the standard complex structure of C^n. We define

$$g_\alpha = g_0\big|_{S^{2n+1}}, J = J_0\big|_{S^{2n+1}},$$

then $(S^{2n+1}, \alpha, g_\alpha, J)$ is an Einstein Sasakian manifold with associated metric g_α. Its contact hyperplane is characterized by

$$Ker\alpha_x = \{Y \in T_x S^{2n+1} | \langle Y, Jx \rangle = 0\}.$$

3 Proof of the theorems

3.1 Several lemmas

In this part we assume that (M, α, g_α, J) is a Sasakian manifold. We show several lemmas which are analogous results of Kähler geometry.

The first lemma shows $\omega = d\alpha$ when restricted to the contact hyperplane $Ker\alpha$ behaviors as the Käkler form on a Käkler manifold.

Lemma 3.1. *Let $X, Y, Z \in Ker\alpha$, then*

$$\bar{\nabla}_X \omega(Y, Z) = 0, \tag{3.1}$$

where $\bar{\nabla}$ is the derivative w.r.t g_α.

Proof.

$$
\begin{aligned}
\bar{\nabla}_X \omega(Y, Z) &= X(\omega(Y,Z)) - \omega(\bar{\nabla}_X Y, Z) - \omega(Y, \bar{\nabla} Z) \\
&= -X g_\alpha(Y, JZ) - \omega(\bar{\nabla}_X Y, Z) - \omega(Y, \bar{\nabla} Z) \\
&= -g_\alpha(\bar{\nabla}_X Y, JZ) - g_\alpha(Y, \bar{\nabla}_X JZ) + g_\alpha(\bar{\nabla}_X Y, JZ) + g_\alpha(Y, J\bar{\nabla}_X Z) \\
&= 0,
\end{aligned}
$$

where in the third equality we used $g_\alpha(Y, \bar{\nabla}_X JZ) = g_\alpha(Y, J\bar{\nabla}_X Z)$, which is a direct corollary of (2.3). $\qquad\square$

Now let L be a Legendrian submanifold of M. We have a nature identification of $NL \cap Ker\alpha$ with T^*L, where NL is the normal bundle of L and T^*L is the cotangent bundle.

Definition 3.2. $\tilde{\omega} : NL \cap Ker\alpha \mapsto T^*L$ *is the bundle isomorphism defined by*

$$
\tilde{\omega}_p(v_p) = (v_p \lrcorner \omega_p)|_{T_p L},
$$

where $p \in L$ and $v_p \in (NL \cap Ker\alpha)_p$.

Recall that $\omega(\mathbf{R}) = 0$ and $g_\alpha(V, W) = \omega(V, JW)$ for any $V, W \in \xi$, hence $\tilde{\omega}$ defines an isomorphism.

We have

Lemma 3.3. *Let $V \in \Gamma(NL \cap Ker\alpha)$. Then*

$$
\begin{aligned}
\tilde{\omega}(\Delta^\nu V) &= \Delta(\tilde{\omega}(V)) \quad i.e. \\
\Delta^\nu V \lrcorner \omega &= \Delta(V \lrcorner \omega),
\end{aligned}
\tag{3.2}
$$

where Δ is the Laplace-Beltrami operator on (L, g).

Remark 3.4. *This lemma in the context of symplectic geometry was proved by Oh ([Oh90], lemma 3.3). Our proof just follows his argument with only slight modifications.*

Proof. We first show that

$$
\nabla_X(\tilde{\omega}(V)) = \tilde{\omega}(\nabla^\nu_X V)
\tag{3.3}
$$

for any $X \in TL$. Equality (3.3) is equivalent to

$$
\nabla_X(\tilde{\omega}(V))(Y) = \tilde{\omega}(\nabla^\nu_X V)(Y)
\tag{3.4}
$$

for any $Y \in TL$.

$$
\begin{aligned}
\nabla_X(\tilde{\omega}(V))(Y) &= \nabla_X(\tilde{\omega}(V)(Y)) - \tilde{\omega}(V)(\nabla_X Y) \\
&= \bar{\nabla}_X(\omega(V,Y)) - \tilde{\omega}(V)(\nabla_X Y) \\
&= \omega(\nabla^\nu_X V, Y) + \omega(V, \nabla_X Y) - \omega(V, \nabla_X Y) \\
&= \omega(\nabla^\nu_X V, Y) \\
&= \tilde{\omega}(\nabla^\nu_X V)(Y).
\end{aligned}
$$

10

For the third equality we used $\bar{\nabla}_X \omega = 0$, when restricted to $Ker\alpha$, which is proved in lemma 3.1.

Let $p \in L$ and we choose an orthogonal frame $\{E_1, ..., E_n\}$ on TL with $\nabla_{E_i} E_j(p) = 0$, then the general Laplacian Δ can be written as

$$\Delta \psi(p) = \sum_{i=1}^{n} \nabla_{E_i} \nabla_{E_i} \psi(p),$$

where ψ is a tensor on L. Therefore

$$
\begin{aligned}
(\tilde{\omega}^{-1} \circ \Delta \cdot \tilde{\omega}(V))(p) &= (\tilde{\omega}^{-1} \circ \sum_{i=1}^{n} \nabla_{E_i} \nabla_{E_i} \tilde{\omega}(V))(p) \\
&= \sum_{i=1}^{n} (\tilde{\omega}^{-1} \nabla_{E_i} \tilde{\omega} \cdot \tilde{\omega}^{-1} \nabla_{E_i} \tilde{\omega}(V))(p) \\
&= \sum_{i=1}^{n} \nabla_{E_i}^{\nu} \nabla_{E_i}^{\nu} V(p) \\
&= (\Delta^{\nu} V)(p),
\end{aligned}
$$

where in the third equality we used (3.4). $\qquad\square$

3.2   Proof of theorem 1.1

We see that for any function s defined on L,

$$
\begin{aligned}
0 &= \int_L s \, div JH d\mu \\
&= \int_L g(JH, \nabla s) d\mu \\
&= \int_L \omega(H, \nabla s) d\mu \\
&= \int_L \langle \omega \rfloor H, \omega \rfloor \nabla s \rangle d\mu \\
&= \int_L \langle \omega \rfloor H, ds \rangle \\
&= \int_L \delta(\omega \rfloor H) s d\mu.
\end{aligned}
$$

Therefore the E-L equation for L is equivalent to

$$\delta(\omega \rfloor H) = 0, \tag{3.5}$$

where δ is the adjoint operator of d on L.

By lemma 2.7 we see that L satisfies

$$\Delta_h(\omega \rfloor H) = 0, \tag{3.6}$$

11

where $\Delta_h := \delta d + d\delta$ is the Hodge-Laplace operator. That is the mean curvature form of L is a harmonic one form.

To proceed on, we need the following Weitzenböck formula

Lemma 3.5. *Let M be an n dimensional oriented Riemannian manifold. If $\{V_i\}$ is a local orthonormal frame field and $\{\omega^i\}$ is its dual co-frame field, then*

$$\Delta_h = -\sum_i D^2_{V_i V_i} + \sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j},$$

where $D^2_{XY} \equiv D_X D_Y - D_{D_X Y}$ represents the covariant derivatives, $\Delta_d = d\delta + \delta d$ is the Hodge-Laplace and $R_{XY} = -D_x D_Y + D_Y D_X + D_{[X,Y]}$ is the curvature tensor.

For a detailed discussion on the Weitzenböck formula we refer to Wu ([Wu]).

Using the Weitzenböck formula we have

$$-\Delta(\omega \lfloor H) + \sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j}(\omega \lfloor H) = 0, \tag{3.7}$$

where $\{V_i\}$ is a local orthogonal frame field and $\{\omega^i\}$ is its dual co-frame field on L.

Denote $\omega \lfloor H$ by $\theta_H = \sum_k \theta_k \omega^k$, we have

$$
\begin{aligned}
\sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j} \theta_H &= \sum_{ij} R_{V_i V_j} \theta_H(V_j) \omega^i \\
&= \sum_{ijk} R_{V_i V_j} \omega^k(V_j) \theta_k \omega^i \\
&= -\sum_{ijk} \omega^k(R_{V_i V_j} V_j) \theta_k \omega^i \\
&= -\sum_{ijk} \langle R_{V_i V_j} V_j, V_k \rangle \theta_k \omega^i \\
&= -\sum_{ij} \langle R_{V_i V_j} V_j, V_i \rangle \theta_i \omega^i \\
&= K \theta_H.
\end{aligned}
$$

That is

$$\sum_{ij} \omega^i \wedge i(V_j) R_{V_i V_j}(\omega \lfloor H) = K \omega \lfloor H. \tag{3.8}$$

Recall that $H \in NL \cap Ker\alpha$, using (3.3) to H we get

$$\Delta(\omega \lfloor H) = \Delta^\nu H \lfloor \omega. \tag{3.9}$$

Combing (3.7), (3.8) and (3.9) we have

$$
\begin{aligned}
0 &= -\Delta^\nu H \lfloor \omega + K \omega \lfloor H \\
&= (-\Delta^\nu H + KH) \lfloor \omega,
\end{aligned}
$$

which implies that

$$-\Delta^\nu H + KH = f\mathbf{R} \tag{3.10}$$

for some function f on L.

The next lemma is one of our key observations which states that a Legendrian submanifold in a Sasakian manifold is contact stationary if and only if $\langle \Delta^\nu H, \mathbf{R} \rangle = 0$.

Lemma 3.6. *Let $L \subseteq (M^{2n+1}, \alpha, g_\alpha, J)$ be a contact Legendrian stationary submanifold. Then we have $\Delta^\nu H$ is orthogonal to $\mathbf{R}$.*

Proof. For any point $p \in L$, we choose a local orthonormal frame $\{E_i : i = 1, ..., n\}$ of L such that $\nabla_{E_i} E_j(p) = 0$. We have at p (in the following computation we adopt the Einstein summation convention)

$$
\begin{aligned}
\langle \Delta^\nu H, \mathbf{R} \rangle &= \sum_i \langle \nabla^\nu_{E_i} \nabla^\nu_{E_i} H, \mathbf{R} \rangle \\
&= E_i \langle \nabla^\nu_{E_i} H, \mathbf{R} \rangle - \langle \nabla^\nu_{E_i} H, \bar\nabla_{E_i} \mathbf{R} \rangle \\
&= E_i \langle \nabla^\nu_{E_i} H, \mathbf{R} \rangle + \langle \nabla^\nu_{E_i} H, JE_i \rangle \\
&= E_i (E_i \langle H, \mathbf{R} \rangle - \langle H, \bar\nabla_{E_i} \mathbf{R} \rangle) + \langle \nabla^\nu_{E_i} H, JE_i \rangle \\
&= E_i \langle H, JE_i \rangle + \langle \nabla^\nu_{E_i} H, JE_i \rangle \\
&= 2\langle \nabla^\nu_{E_i} H, JE_i \rangle + \langle H, \bar\nabla_{E_i} JE_i \rangle \\
&= 2\langle \nabla^\nu_{E_i} H, JE_i \rangle + \langle H, J\bar\nabla_{E_i} E_i \rangle \\
&= 2\langle \nabla^\nu_{E_i} H, JE_i \rangle \\
&= 2\langle \bar\nabla_{E_i} H, JE_i \rangle \\
&= -2\langle J\bar\nabla_{E_i} H, E_i \rangle \\
&= -2\langle \bar\nabla_{E_i} JH, E_i \rangle \\
&= -2\langle \nabla_{E_i} JH, E_i \rangle \\
&= -2 div_g(JH) \\
&= 0.
\end{aligned}
$$

Note that in this computation we used lemma 2.3, lemma 2.8 and lemma 2.9 several times and the last equality holds because L is contact stationary. $\quad\square$

Therefore we have $-\Delta^\nu H + KH \perp \mathbf{R}$ by this lemma and lemma 2.8, which shows $f \equiv 0$, i.e.

$$-\Delta^\nu H + KH = 0,$$

and we are done. $\quad\square$

3.3 Proof of theorem 1.6

Let L be a Legendrian surface in S^5 with the induced metric g. Let $\{e_1, e_2\}$ be an orthogonal frame on L such that $\{e_1, e_2, Je_1, Je_2, \mathbf{R}\}$ be a orthogonal frame on S^5.

In the following we use indexes i, j, k, l, s, t, m and β, γ such that

$$
\begin{aligned}
1 \leq i, j, k, l, s, t, m &\leq 2, \\
1 \leq \beta, \gamma &\leq 3, \\
\gamma^* = \gamma + 2, \quad \beta^* &= \beta + 2.
\end{aligned}
$$

Let B be the second fundamental form of L in S^5 and define

$$
\begin{aligned}
h_{ij}^k &= g_\alpha(B(e_i, e_j), Je_k), & (3.11) \\
h_{ij}^3 &= g_\alpha(B(e_i, e_j), \mathbf{R}). & (3.12)
\end{aligned}
$$

Then

$$
\begin{aligned}
h_{ij}^k &= h_{ik}^j = h_{kj}^i, & (3.13) \\
h_{ij}^3 &= 0. & (3.14)
\end{aligned}
$$

The Gauss equations and Ricci equations are

$$
R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_s (h_{ik}^s h_{jl}^s - h_{il}^s h_{jk}^s) \tag{3.15}
$$

$$
R_{ik} = \delta_{ik} + 2\sum_s H^s h_{ik}^s - \sum_{s,j} h_{ij}^s h_{jk}^s, \tag{3.16}
$$

$$
2K = 2 + 4H^2 - S, \tag{3.17}
$$

$$
\begin{aligned}
R_{3412} &= \sum_i (h_{i1}^1 h_{i2}^2 - h_{i2}^1 h_{i1}^2) \\
&= \det h^1 + \det h^2, \tag{3.18}
\end{aligned}
$$

where h^1, h^2 are the second fundamental forms w.r.t. the directions Je_1, Je_2 respectively.

In addition we have the following Codazzi equations and Ricci identities

$$
h_{ijk}^\beta = h_{ikj}^\beta, \tag{3.19}
$$

$$
h_{ijkl}^\beta - h_{ijlk}^\beta = \sum_m h_{mj}^\beta R_{mikl} + \sum_m h_{mi}^\beta R_{mjkl} + \sum_\gamma h_{ij}^\gamma R_{\gamma^*\beta^*kl}. \tag{3.20}
$$

Using these equations, we can get the following Simons type inequality:

Lemma 3.7. *Let L be a Legendrian surface in S^5. Then we have*

$$
\begin{aligned}
\frac{1}{2}\Delta \sum_{i,j,\beta}(h_{ij}^\beta)^2 \geq\ & |\nabla^T h|^2 - 2|\nabla^T H|^2 - 2|\nabla^\nu H|^2 + \sum_{i,j,k,\beta}(h_{ij}^\beta h_{kki}^\beta)_j \\
& + S - 2H^2 + 2(1 + H^2)\rho^2 - \rho^4 - \frac{1}{2}S^2, \tag{3.21}
\end{aligned}
$$

where $|\nabla^T h|^2 = \sum_{i,j,k,s}(h_{ijk}^s)^2$ and $|\nabla^T H|^2 = \sum_{i,s}(H_i^s)^2$.

Proof. Using equations from (3.15) to (3.20), we have

$$
\begin{aligned}
\frac{1}{2}\Delta \sum_{i,j,\beta}(h_{ij}^\beta)^2 &= \sum_{i,j,k,\beta}(h_{ijk}^\beta)^2 + \sum_{i,j,k,\beta} h_{ij}^\beta h_{kijk}^\beta \\
&= |\nabla h|^2 - 4|\nabla^\nu H|^2 + \sum_{i,j,k,\beta}(h_{ij}^\beta h_{kki}^\beta)_j + \sum_{i,j,l,k,\beta} h_{ij}^\beta(h_{lk}^\beta R_{lijk} + h_{il}^\beta R_{lj}) \\
&\quad + \sum_{i,j,k,l,\beta,\gamma} h_{ij}^\beta h_{ki}^\gamma R_{\gamma^*\beta^* jk} \\
&= |\nabla h|^2 - 4|\nabla^\nu H|^2 + \sum_{i,j,k,s}(h_{ij}^s h_{kki}^s)_j + 2K\rho^2 - 2(\det h^1 + \det h^2)^2 \\
&\geq |\nabla h|^2 - 4|\nabla^\nu H|^2 + \sum_{i,j,k,\beta}(h_{ij}^\beta h_{kki}^\beta)_j + 2(1+H^2)\rho^2 - \rho^4 - \frac{1}{2}S^2, \quad (3.22)
\end{aligned}
$$

where $\rho^2 := S - 2H^2$ and in the above calculations we used the following identities

$$
\begin{aligned}
\sum_{i,j,k,l,\beta} h_{ij}^\beta(h_{lk}^\beta R_{lijk} + h_{il}^\beta R_{lj}) &= 2K\rho^2, \\
\sum_{i,j,k,l,\beta,\gamma} h_{ij}^\beta h_{ki}^\gamma R_{\gamma^*\beta^* jk} &= -2(\det h^1 + \det h^2)^2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|\nabla h|^2 &= \sum_{i,j,k,\beta}(h_{ijk}^\beta)^2 \\
&= |\nabla^T h|^2 + \sum_{i,j,k}(h_{ijk}^3)^2 \\
&= |\nabla^T h|^2 + \sum_{i,j,k}(h_{ij}^k)^2 \\
&= |\nabla^T h|^2 + S, \quad (3.23)
\end{aligned}
$$

where in the third equality we used

$$
\begin{aligned}
h_{ijk}^3 &= \langle \bar{\nabla}_{e_k} B(e_i, e_j), \mathbf{R} \rangle \\
&= -\langle B(e_i, e_j), \bar{\nabla}_{e_k} \mathbf{R} \rangle \\
&= \langle B(e_i, e_j), J e_k \rangle \\
&= h_{ij}^k.
\end{aligned}
$$

Similarly we have

$$
|\nabla^\nu H|^2 = |\nabla^T H|^2 + H^2. \quad (3.24)
$$

Combing (3.22), (3.23) and (3.24) we get (3.21). $\qquad \square$

Now we prove an integral equality for L, by using the equation (1.3).

Lemma 3.8. *Let $L : \Sigma \mapsto S^5$ be a contact stationary Legendrian surface, where S^5 is the unit sphere with standard contact structure and metric. Then*

$$\int_L |\nabla^\nu H|^2 d\mu = -\int_L K H^2 d\mu, \tag{3.25}$$

where $|\nabla^\nu H|^2 = \sum_{\beta,i}(H_i^\beta)^2$.

Proof. By using (1.3) we have

$$
\begin{aligned}
|\nabla^\nu H|^2 &= \sum_{\beta,i}(H_i^\beta)^2 \\
&= \sum_{\beta,i}(H_i^\beta H^\beta)_i - \sum_\beta H^\beta \Delta^\nu H^\beta \\
&= \sum_{\beta,i}(H_i^\beta H^\beta)_i - K H^2. \tag{3.26}
\end{aligned}
$$

We get (3.25) by integrating over (3.26). □

Integrating over (3.21) and using $|\nabla^T h|^2 \geq 3|\nabla^T H|^2$ ([Li02] lemma 2.4) we get

$$
\begin{aligned}
0 &\geq \int_L [(|\nabla^T h|^2 - 2|\nabla^T H|^2) - 2|\nabla^\nu H|^2 + S - 2H^2 + 2(1+H^2)\rho^2 - \rho^4 - \tfrac{1}{2}S^2]d\mu \\
&\geq \int_L [-2|\nabla^\nu H|^2 + S - 2H^2 + 2(1+H^2)\rho^2 - \rho^4 - \tfrac{1}{2}S^2]d\mu \\
&= \int_L (2-\rho^2)\rho^2 d\mu + \int_L 2H^2\rho^2 + 2KH^2 - 2H^2 + S - \tfrac{1}{2}S^2 d\mu \\
&= \int_L (2-\rho^2)\rho^2 d\mu + \int_L 2H^2\rho^2 + (2+4H^2-S)H^2 - 2H^2 + S - \tfrac{1}{2}S^2 d\mu \\
&= \int_L (2-\rho^2)\rho^2 d\mu + \int_L H^2 S + S - \tfrac{1}{2}S^2 d\mu \\
&= \int_L (2-\rho^2)\rho^2 d\mu + \int_L H^2(S-2) + 2H^2 + \tfrac{S}{2}(2-S)d\mu \\
&= \int_L (2-\rho^2)\rho^2 + (2-S)(\tfrac{S}{2} - H^2) + 2H^2 d\mu \\
&= \int_L \rho^2(2-\rho^2) + \tfrac{\rho^2}{2}(2-S) + 2H^2 d\mu \\
&= \int_L \tfrac{3}{2}\rho^2(2-S) + 2H^2\rho^2 + 2H^2 d\mu.
\end{aligned}
$$

This implies the desired integral inequality

$$\int_L \tfrac{3}{2}\rho^2(2-S) + 2H^2\rho^2 + 2H^2 d\mu \leq 0.$$

In addition if $0 \leq S \leq 2$, we must have $\rho^2 = 0$ and $H = 0$ and so $S = 0$, i.e. L is totally geodesic or $S = 2$ and $H = 0$, i.e. L is a flat Legendrian minimal torus. □

3.4 Proof of theorem 1.9

Let M be a suface in S^n with second fundamental form $B = (h_{ij}^\alpha)$, where

$$1 \le i, j \le 2, 3 \le \alpha \le n.$$

We define the trace free tensor

$$\tilde{h}_{ij}^\alpha = h_{ij}^\alpha - H^\alpha g_{ij}, \tag{3.27}$$

then the E-L equation (1.8) for Willmore surfaces becomes

$$\Delta^\nu H^\alpha + \sum_{\beta, i, j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta H^\beta = 0, 3 \le \alpha \le n. \tag{3.28}$$

We have

Lemma 3.9. *Let M be a Willmore surface. Then*

$$\int_M |\nabla^\nu H|^2 d\mu = \int_M \sum_{\alpha, i} (H_i^\alpha)^2 d\mu = \int_M \sum_{\alpha, \beta} \tilde{\sigma}_{\alpha\beta} H^\alpha H^\beta d\mu, \tag{3.29}$$

where $\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}_{ij}^\alpha \tilde{h}_{ij}^\beta$.

Proof. By using (3.28),

$$\begin{aligned}
|\nabla^\nu H|^2 &= \sum_{\alpha, i} (H_i^\alpha)^2 \\
&= \sum_{\alpha, i} (H^\alpha H_i^\alpha)_i - \sum_\alpha H^\alpha \Delta^\nu H^\alpha \\
&= \sum_{\alpha, i} (H^\alpha H_i^\alpha)_i + \sum_{\alpha, \beta} \tilde{\sigma}_{\alpha\beta} H^\alpha H^\beta.
\end{aligned}$$

Integrating over this equality we get (3.29). $\qquad\qquad\square$

Because $(\tilde{\sigma}_{\alpha\beta})$ is a symmetric matrix we can assume that it is diagonal, by choosing appropriate local frame field on L. Hence we can assume that

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_\alpha \delta_{\alpha\beta}. \tag{3.30}$$

Then we see that

$$\rho^2 = \sum_\alpha \tilde{\sigma}_\alpha. \tag{3.31}$$

Now assume that L is a Legendrian Willmore surface in S^5. In the proof of (3.21) we see that

$$\begin{aligned}
\frac{1}{2} \Delta \sum_{i,j,\beta} (h_{ij}^\beta)^2 &= |\nabla h|^2 - 4|\nabla^\nu H|^2 + \sum_{i,j,k,s} (h_{ij}^s h_{kki}^s)_j + 2K\rho^2 - 2(\det h^1 + \det h^2)^2 \\
&\ge |\nabla^T h|^2 - 4|\nabla^\nu H|^2 + \sum_{i,j,k,\beta} (h_{ij}^\beta h_{kki}^\beta)_j \\
&\quad + S + 2(1 + H^2)\rho^2 - \rho^4 - \frac{1}{2} S^2. \tag{3.32}
\end{aligned}$$

Integrating over this inequality for L we have

$$
\begin{aligned}
0 \;\geq\; & \int_L |\nabla^T h|^2 - 4|\nabla^\nu H| + 2H^2\rho^2 + \rho^2(2-\rho^2) + S - \frac{1}{2}S^2 d\mu \\
\geq\; & \int_L -4|\nabla^\nu H| + 2H^2\rho^2 + \rho^2(2-\rho^2) + S - \frac{1}{2}S^2 d\mu \\
=\; & \int_L -4|\nabla^\nu H| + 4H^2\rho^2 + \rho^2(2-S) + \frac{S}{2}(2-S)d\mu \\
=\; & \int_L \rho^2(2-S) + \frac{S}{2}(2-S)d\mu + 4\int_L H^2\rho^2 - \sum_{\alpha\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}d\mu \\
=\; & \int_L (\rho^2 + \frac{S}{2})(2-S)d\mu + 4\int_L H^2\rho^2 - \sum_{\alpha\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta}d\mu. \qquad (3.33)
\end{aligned}
$$

Noting that

$$
H^2\rho^2 = (\sum_\alpha (H^\alpha)^2)(\sum_\beta \tilde{\sigma}_\beta) \geq \sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha = \sum_{\alpha\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta},
$$

we get

$$
\int_L (\rho^2 + \frac{S}{2})(2-S)d\mu \leq 0. \qquad (3.34)
$$

In particular, if

$$
0 \leq S \leq 2,
$$

then either $S = 0$, i.e. L is totally geodesic or $S = 2$.

Now let us analysis the case $S = 2$. In this case all of the inequalities in (3.32) and in (3.33) will be equalities. In particular we have

$$
(\det h^1 + \det h^2)^2 = \frac{S^2}{4},
$$

which implies that $\det h^1 + \det h^2 = \frac{S}{2}$ or $-\frac{S}{2}$.

If $\det h^1 + \det h^2 = \frac{S}{2}$, we see that

$$
\rho^2 = S - 2H^2 = \frac{S}{2} - (\det h^1 + \det h^2) = 0.
$$

Furthermore by the integral equality (3.29) we have

$$
0 \leq \int_L |\nabla^\nu H|^2 d\mu = \int_M \sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta} H^\alpha H^\beta d\mu \leq \int_L H^2\rho^2 d\mu = 0,
$$

which implies that $|\nabla^\nu H|^2 = 0$.

Recall that we proved

$$
|\nabla^\nu H|^2 = |\nabla^T H|^2 + H^2,
$$

and so we have $H = 0$. Hence finally we get $\rho^2 = H^2 = 0$, which shows $S = 0$, a contradiction.

Therefore we must have $\det h^1 + \det h^2 = -\frac{S}{2}$. Noting that in this case

$$\rho^2 = S - 2H^2 = \frac{S}{2} - (\det h^1 + \det h^2) = S,$$

we have $H^2 = 0$, i.e. L is a minimal surface. Therefore in this case L is a flat Legendrian minimal torus.

$\square$

References

[Blair] D. E. Blair, Reimannian geometry of contact and symplectic manifolds, *Progress in Math., vol 203*(2002), Birkhäuser, Basel.

[Bu] A. Butscher, Equivariant gluing construction of contact-stationary Legendrian submanifolds in the $(2n+1)$-sphere, *Cal. Var. and Partial Differential Equations 35 (2009), no.1*, 57-102.

[BKSS] K. Benko, M. Kothe, K. D. Semmler and U. Simon, Eigenvalues of the Laplacian and curvature, *Colloq. Math.* **42** (1979), 19-31.

[C-Li-U] Ildefonso Castro, Haizhong Li and Francisco Urbano, Hamiltonian-minimal lagrangian submanfolds in complex space form, *Pacific Journal of Mathematics* **227**(2006), 43-65.

[C-C-K] S. S. Chern, M. Do. Carmo and S. Kobayashi, Minimal furfaces of a sphere with second fundamental form of constant length, *in Functional Analysis and Related Fields*, F. Brower, ed., Springer-Verlag, Berlin, 1970, pp. 59-75.

[KoSi] M. Kozlowski and U. Simon, Minimal immersions of 2-manifolds into spheres, *Math. Z.* **186**(1984), 377-382.

[Law] H. B. Lawson, Local rigidity theorems of minimal hypersurfaces, *Ann. Of Math.* **89**(1969), 237-266.

[Le] H. V. Lê, A minimizing deformation of Legendrian submanifolds in the standard sphere, *Diff. Geom. and its Appl.* **21**, 2004, 297-316.

[DV] F. Dillen and L. Vrancken, C-totally real submanifolds of Sasakian space forms, *J. Math. Pures Appl.* **69**(1990), 85-93.

[Ha] M. Haskins, Special Lagrangian cones, *Amer. Journal of Math.* **126**(2004), 845-871.

[HR] Frédéric Hélein and Pascal Romon, Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces, Differential geometry and integrable systems (Tokyo, 2000), *Contemp. Math., vol. 308, Amer. Math. Soc.*, Providence, RI, 2002, 161-178.

[Li01] H. Z. Li, Willmore hypersurfaces in a sphere, *Asian J. Math.* **5**(2001), 365-378.

[Li02] H. Z. Li, Willmore Surfaces in S^n, Ann. Glob. Anal. Geom., **21**, 203-213, 2002.

[LOY] G. D. Ludden, M. Okumura and K. Yano, A totally real surface in CP^2, *Proc. Amer. Math. Soc.***53**(1975), 186-190.

[Oh90] Y.-G. Oh, Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds, *Invent. Math.* **101**(2), 1990, 501-519.

[Oh93] Y.-G. Oh: Volumn minimization of Lagrangain submanifolds under Hamiltonian deformations, *Math. Z.* **212**(2), 1993, 175-192.

[Rec] H. Reckziegel, A correspondence between horizontal submanifolds of Sasakian manifolds and totally real submanifolds of Kählerian manifolds, *Topics in Differential Geometry,* vols. I, II(Debrecen, 1984), North-Holland, Amsterdam, 1988, pp. 1063-1081.

[Siom] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. Of Math.* **88**(1968), 62-105.

[Smoc] K. Smoczyk, Colsed Legendre geodesics in Sasaki manifolds, *New York J. Math.***9**(2003), 23-47.

[ScW] R. Schoen and J. Wolfson: Minimizing area among Lagrangian surfaces: the mapping problem, *J. Diff. Geom,* **58**, 1-86, 2001.

[Weiner] J. Weiner, On a problem of Chen, Willmore, et al., *Indiana Univ. Math. J.* **27**(1978), 19-35.

[Wu] H. Wu, The Bochner technique in differential geometry, *Mathematical Reports, Volume 3, part 2*(1988), Harwood Academic Publishers, London-Pairs.

[Yau] Shing-Tung Yau, Submanifolds with constant mean curvature I, *Amer. J. of Math.* **96**(1974), 346-366.

[YKM] S. Yamaguchi, M. Kon and Y. Miyahara, A theorem on C-totally real minimal surface, *Proc. Amer. Math. Soc.***54**(1976), 276-280.

YONG LUO
MATHEMATISCHES INSTITUT,
ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG,
ECKERSTR. 1, 79104 FREIBURG, GERMANY.
yong.luo@math.uni-freiburg.de